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Symmetries of String Theory in $AdS_5 \times S^5$

MADOKA NISHIMURA

Department of Community Service and Science

Tohoku University of Community Service and Science

3-5-1 Iimoriyama, Sakata, Yamagata 998-8580, Japan

Abstract

The isometry transformations of the type IIB superstring theory in a five-dimensional anti de Sitter (AdS) spacetime with a five-sphere are discussed. The string theory on AdS spacetime is useful to examine a duality between open and closed strings.

概　　要

数学的な興味から研究され続けてきた超弦理論は、現在、素粒子物理学や宇宙物理学における実験・観測結果の説明に使われ始めている。この超弦理論の誕生から現在までの発展と、非摂動論的な超弦理論がもつ双対性から導き出される予想（AdS／CFT対応）について紹介する。また、超弦理論における開弦・閉弦双対性を理解するために重要な、反ド・ジッター時空上の超弦理論を考え、そのアイソメトリー対称性を議論する。

1. Introduction

The string theory [1, 2] is a theory which describes all the interactions of the fundamental particles in nature. Several string theories with different types of gauge symmetries were constructed in the past. It is now believed that these string theories are different aspects of one fundamental theory, a mother theory called the M-theory. A main issue in the research of the string theory had been to understand a mathematical structure of the theory. However, it now becomes more attractive to particles physicists after a discovery of non-perturbative objects called D-branes [3] in it. It is applicable to phenomenology of realistic models of particle physics and cosmology. In future the string theory may be tested in accelerator physics or in observations in the universe.

In this paper I would like to discuss a certain formulation of a string theory in a curved spacetime. In the rest of this section I will first summarize a history of the string theory starting from the birth of the string theory in the strong interaction physics to recent developments. Then, I will explain what I would like to discuss in this paper.

1.1 History of string theory

The string theory was originally constructed in the late '60s as a model of the strong interaction, i.e. an interaction among quarks in unclei and other hadrons. A meson made of a quark and an anti-quark behaves like a tube or a string when they are separated. The string model of hadrons can explain particles with large spins, which had been difficult to explain in older models of the strong interaction. Until a new theory, quantum chromodynamics (QCD) was found in the mid '70s, the string model had been a successful theory of the strong interaction. Particle physicists suspected that the string model was too mathematical and restrictive by many theorems so that it does not have enough freedom to explain more complex patterns of scattering processes in accelerator experiments. Theoretical physicists, however, studied the string model adoring its mathematical beauty.

After the invention of QCD the original string model for the strong interaction was discarded by particle physicists quite for a while. The string theory was revived in the mid '80s as a unified theory of all the interactions of the fundamental particles. Since then, the string theory has been extensively studied by huge numbers of particle and theoretical physicists. These studies are mainly concerned with mathematical structures of the string theory rather than applications to realistic models of particle physics.

A string theory in general is a field theory on a two-dimensional world-sheet embedded in spacetime, which a string sweeps out. There are two kinds of strings:closed strings and open strings. In a perturbative formulation of the string theory these two kinds of strings are distinguished by different boundary conditions on the world-sheet. One can also consider string theories with spacetime supersymmetry, a symmetry between bosons and fermions. Before 1995 five supersymmetric string theories were known: type I, type IIA, type IIB, SO (32) heterotic and $E_8 \times E^8$ heterotic string theories. The type I theory contains both of open and closed strings, while other four theories contain only closed strings.

1.2 Recent developments in string theory

In 1995 it was discovered that non-perturbative objects, D-branes, exist as constituents of string theories. Using D-branes non-perturbative aspects of string theories have begun to be understood. Important issues in field theories of particle physics often need analysis in a strong coupling region, where non-perturbative effects are crucial. Understanding of its non-perturbative effects have turned back the string theory to applications in various phenomena in nature.

A useful concept in non-perturbative analysis of field theories is duality, which means an equivalence between two theories which look quite different. The duality often exchanges an elementary excitation of a field in one theory and an excitation on a non-perturbative object such as a soliton in another theory. A non-perturbative object in the string theory was found from open strings with the Dirichlet boundary condition, and is called D(irichlet)-brane [3]. Detailed studies of a relation between a field theory on the D-branes and a geometry around the D-branes showed that there is a duality between them. As the field theory is in general an effective theory of open strings while the geometry is represented by a supergravity which is an effective theory of closed strings, this is open-closed duality. The simplest D-brane in flat spacetime looks like a board and is infinitely extended. Some other simple configurations of D-branes are known and were classified. These various configurations of D-branes have interpretations in terms of field theories and also in terms of supergravities, and there are dualities between them.

The most well understood cases of these dualities are the ones between supersymmetric conformal field theories (CFTs) and supergravities (or closed superstring theories) in anti de Sitter (AdS) spacetime with a certain compact space. In particular, the $N=4$ super Yang-Mills theory in four-dimensional spacetime and the type IIB supergravity in a five-dimensional AdS spacetime with a five-sphere ($AdS_5 \times S^5$) is studied in detail. These dualities are called the AdS/CFT correspondence [4]-[7]. In the AdS/CFT correspondence the number of D-branes must be large. This means that the number of colors N is large on the field theory side and quantum effects are small on the supergravity side. Furthermore, the strong coupling limit on the field theory sides corresponds to small stringy effects on the supergravity side. Thus, the strong coupling region of a field theory in large N limit can be studied by using a classical supergravity.

A problem in the AdS/CFT correspondence is that the field theory needs to have the conformal symmetry, which is not realized in the real world. The conformal symmetry on the field theory side comes from the isometry of AdS spacetime on the supergravity side. To apply the AdS/CFT correspondence to real phenomena one needs to get out from the conformal symmetry. A new geometry which has a lower symmetry should be found.

1.3 String theory in AdS spacetime

To understand the AdS/CFT correspondence more fully one needs to study a string theory rather than its effective theory, supergravity, in the AdS spacetime. The string theory in a curved spacetime has an action with a curved metric and other fields such as antisymmetric tensor fields as backgrounds. To introduce background fields in the Ramond-Ramond sector it is easier to use the Green-Schwarz formalism of the superstring theory. The $AdS_5 \times S^5$ background we are interested in has a five-form background in the Ramond-Ramond sector. The action of the string theory in

$AdS_5 \times S^5$ was constructed using the Green-Schwarz formalism in refs. [8]-[10]. The gauge-fixed action was then obtained in refs. [11]-[14]. Although the action of the string theory in $AdS_5 \times S^5$ was obtained, the symmetry structure is not yet discussed sufficiently. Here I would like to obtain the symmetry transformations explicitly. The explicit form of the symmetry transformations is required when one explicitly. The explicit form of the symmetry transformations is required when one extends the AdS/CFT correspondence to more general situations.

One of the problems in the AdS/CFT correspondence is that needs large number of colors, whereas nature has only three. It also needs a strong coupling limit on the field theory side. In refs. [15, 16] a background which does not need these limit is used to study the AdS/CFT correspondence. The string theory in this background is still in the AdS geometry, but strings are spinning on the sphere. This background is called the pp-wave background. Correlation functions of operators in the CFT can be calculated in terms of an integrable system or a spin chain system well-known in the condensed matter physics. However, only a subsector of the whole system is considered in these studies. Contributions of fermionic degrees of freedom are also important but not fully understood. To understand these issues one has to know the symmetry structure of the string theory in $AdS_5 \times S_5$.

In the following sections we would like to study the symmetry structure of the string theory in $AdS_5 \times S^5$. In the next section we review a geometry of AdS spacetime and its isometry. In sect.3 we discuss the Green-Schwarz type action of the type IIB string theory in $AdS_5 \times S^5$. We will obtain an explicit form of a gauge-fixed action. In sect.4 we will examine symmetry transformations of the gauge-fixed action. All the symmetry transformations except those corresponding to special conformal transformations and conformal supertransformations on the CFT side are obtained. We will discuss how to obtain these last two transformations in sect.5.

2. AdS$_{p+2}$ and its isometry

Anti de Sitter (AdS) spacetime is a maximally symmetric spacetime which has a negative constant curvature. It is convenient to represent $(p+2)$-dimensional AdS (AdS$_{p+2}$) specetime as a subspace embedded in $\mathbf{R}^{p+3}$ with coordinates $X^{\hat{M}}(\hat{M}=0, 1, \cdots, p+2)$ by

$$-\left(X^0\right)^2+\sum_{i=1}^{p+1}\left(X^i\right)^2-\left(X^{p+2}\right)^2=-R^2, \tag{2.1}$$

where the constant R is called a radius. $X^{\hat{M}}$ satisfying this equation can be parametrized by the $(p+2)$-dimensional independent coordinates $x^M=(x^m, x^{p+1}=r),(m=0,\cdots,p)$ as

$$X^m=Rrx^m \quad (m=0, 1,\cdots,p),$$

$$X^{p+1}=\frac{1}{2r}\left[1-r^2\left(R^2-x^m x^n \eta_{mn}\right)\right],$$

$$X^{p+2}=\frac{1}{2r}\left[1-r^2\left(R^2-x^m x^n \eta_{mn}\right)\right], \tag{2.2}$$

where $\eta_{mm}=\mathrm{diag}(-1, 1, \cdots, 1)$ is the $(p+1)$-dimensional flat metric. In terms of these coordinates the metric of AdS spacetime is given by

$$ds^2=-\left(dX^0\right)^2+\sum_{i=1}^{p+1}\left(dX^i\right)^2-\left(dX^{p+2}\right)^2$$

$$=R^2\left(r^2dx^m dx^n \eta_{mn}+\frac{dr^2}{r^2}\right)\equiv dx^M dx^N g_{MN}(x). \tag{2.3}$$

To introduce spinors we need to use a vielbein $e_M{}^A(A=0, \cdots, p+1)$, which is related to the metric as $g_{MN}=e_M{}^A e_M{}^B\ \eta_{AB}$ with $\eta_{AB}=\mathrm{diag}\,(-1,+1,\cdots,+1)$. We choose the vielbein as

$$e^a=dx^m\,\delta_m^a r \qquad e^{p+1}=\frac{dr}{r}. \tag{2.4}$$

Then, non-zero components of the spin connection becomes

$$\omega^{ap+1}=dx^m\,\delta_m^a r. \tag{2.5}$$

From eq. (2.1) it is obvious that the AdS spacetime has an isometry SO(2, p+1). An infinitesimal SO(2, $p+1$) transformation is

$$\delta X^{\hat{M}}=a^{\hat{M}}{}_{\hat{N}}X^{\hat{N}}, \tag{2.6}$$

where $a^{\hat{M}}{}_{\hat{N}}$ are constant transformation parameters satisfying $a^{\hat{M}\hat{N}}\equiv a^{\hat{M}}{}_{\hat{P}}\ \eta^{\hat{P}\hat{N}}=-a^{\hat{M}\hat{N}}(\eta^{\hat{M}\hat{N}}\ =\mathrm{diag}\,(-1,+1,\cdots,+1,-1))$. In terms of the independent coordinates this transformation becomes

$$\begin{aligned}
&(\mathrm{i})\quad \delta x^m=a^m{}_n x^n,\ \delta r=0,\\
&(\mathrm{ii})\quad \delta x^m=\Lambda x^m,\ \delta r=-\Lambda r,\\
&(\mathrm{iii})\quad \delta x^m=b^m,\ \delta r=0,\\
&(\mathrm{iv})\quad \delta x^m=\zeta^n\left(\frac{\delta^m_n}{r^2}+\delta^m_n x^l x_l-2x^m x_n\right),\ \delta r=2\zeta^m x_m r,
\end{aligned} \tag{2.7}$$

where $x_m=\eta_{mn}x^n$. The parameters Λ, b^m and ζ^m in eq. (2.7) are related to $a^{\hat{M}}{}_{\hat{N}}$ in eq. (2.6) as

$$\Lambda=a^+{}_+,\qquad b^m=-\frac{R}{\sqrt{2}}\,a^m{}_-,\qquad \zeta^m=\frac{1}{\sqrt{2}\mathrm{R}}\,a^m{}_+, \tag{2.8}$$

where $\pm$ denote the directions $X^{\pm}=\frac{1}{\sqrt{2}}(X^{p+1}\pm X^{p+2})$. If we denote all the trans-formations in eq. (2.7) as $\delta x^M=\xi^M(x)$ $(M=0,1,\cdots,P+1)$, it is easy to see that ξ^M is the Killing vector which satisfies

$$D_M\xi_N+\mathrm{D}_N\xi_M=0. \tag{2.9}$$

In the limit $r\to\infty$ the transformations (i), (ii), (iii) and (iv) act on x^m as a Lorentz transformation, a scale transformation, a translation and a special conformal transformation respectively. They together form the conformal group in $p+1$ dimensions, which is isomorphic to the isometry group of ($p+2$)-dimensional AdS spacetime SO (2, $p+1$).

The isometry transformations with transformation parameters ξ^M satisfying eq. (2.9) preserve the form of the metric (2.3). However, when the isometry trans-formations act on the vielbein $e_M{}^A$, they must be accompanied by local Lorentz transformations to preserve the form (2.4). Therefore, the isometry transformations act on the vielbein as

$$\delta e_M{}^A=-\xi^N\partial_N e_M{}^A-\partial_M\xi^N e_M{}^A+\lambda^A{}_B e_M{}^B \tag{2.10}$$

where the parameters of the local Lorentz tranformation $\lambda^A{}_B$ should be determined so that $\delta e_M{}^A = 0$ and are given by

$$\lambda_{AB} = -\xi^N \omega_{NAB} - \frac{1}{2}(D_A \xi_B - D_B \xi_A). \tag{2.11}$$

Using eqs. (2.4), (2.5) we obtain a more explicit form

$$\lambda_{ab} = -\frac{1}{2}(\delta_a^m \eta_{nb} - \delta_b^m \eta_{na})\, \partial_m \xi^n,$$

$$\lambda_{ap+1} = -\frac{1}{2r^2}\, \delta_a^m\, \partial_m \xi^r + \frac{1}{2} r^2 \eta_{na} \partial_r \xi^n. \tag{2.11}$$

For the transformations in eq. (2.7) the parameters ξ^M and $\lambda^A{}_B$ then become

(i) $\xi^m = a^m{}_n x^n,\ \xi^r = 0,\ \lambda_{ab} = a_{ab},\ \lambda_{ap+1} = 0,$

(ii) $\xi^m = \Lambda x^m,\ \xi^r = -\Lambda_r,\ \lambda_{ab} = 0,\ \lambda_{ap+1} = 0,$

(iii) $\xi^m = b^m,\ \xi^r = 0,\ \lambda_{ab} = 0,\ \lambda_{ap+1} = 0,$

(iv) $\xi^m = \zeta^m\left(\frac{1}{r^2} + x^l x_l\right) - 2\zeta^n x_n x^m,\ \zeta^r = 2r\zeta_m x^m,$

$$\lambda_{ab} = -2(\eta_{am}\eta_{bn} - \eta_{bm}\eta_{an})\zeta^n x^m, \lambda_{ap+1} = -\frac{2}{r}\eta_{an}\zeta^n. \tag{2.13}$$

When the isometry transformations act on quantities with local Lorentz indices, we should use a combination of the general coordinate transformations and the local Lorentz transformations with the parameters $\xi^M, \lambda^A{}_B$ in eq. (2.13).

3. String action in $AdS_5 \times S^5$

Let us consider the type IIB string theory in $AdS_5 \times S^5$. The metric of the $AdS_5 \times S^5$ is

$$ds^2 = R^2\left(r^2 dx^m dx^n \eta_{mn} + \frac{dr^2}{r^2} + dx^{m'} dx^{n'} g_{m'n'}\right), \tag{3.1}$$

where the first two terms are the metric of AdS_5 in eq. (2.3) with $p = 3$ and the last term is that of S^5. In the following we put $R = 1$ for simplicity. Our index conventions are as follows. Curved indices are denoted as $\hat{m} = (M, m')$,($M=(m, 4)=(0, \cdots, 4)$, $a' = 5, \cdots, 9$), whereas local Lorentz indices as $\hat{a} = (A, a')$,($A = (a, 4) = (0, \cdots, 4)$, $a' = 5, \cdots, 9$). The primed indices m', a' denote the coordinates of S^5, while the indices M, A denote those of AdS_5. The vielbein and the spin connection are given by (2.4), (2.5) with $p=3$ for AdS_5 and

$$e^{a'}=dx^{m'}e_{m'}{}^{a'},\quad \omega^{a'b'}=dx^{m'}\omega_{m'a'b'} \tag{3.2}$$

for S^5.

It is convenient to represent the ten-dimensional 32×32 gamma matrices $\Gamma^{\hat{a}}$ as

$$\Gamma^{A}=\gamma^{A}\otimes 1\otimes\sigma_1,$$
$$\Gamma^{a'}=1\otimes\gamma^{a'}\otimes\sigma_2, \tag{3.3}$$

where σ_1, σ_2 are the 2×2 Pauli matrices, and γ^{A} and $\gamma^{a'}$ are the 4×4 gamma matrices of SO (1, 4) and SO (5) respectively satisfying

$$\{\gamma^{A},\gamma^{B}\}=2\eta^{AB},\qquad \{\gamma^{a'},\gamma^{b'}\}=2\delta^{a'b'}. \tag{3.4}$$

In this representation a positive chirality spinor Ψ has components

$$\Psi=\begin{pmatrix}\psi\\ 0\end{pmatrix}, \tag{3.5}$$

where ψ has 16 components. The last factors of the gamma matrices in eq. (3.3) act on the two components of eq. (3.5)

The Green-Schwarz type action of the type IIB string theory in $AdS_5\times S^5$ was constructed in refs. [8, 9, 10]. The world-sheet variables are a world-sheet metric $g_{ij}(\sigma)$ and spacetime superspace coordinates $(x^{\hat{m}}(\sigma),\ \Theta^{I}(\sigma))$, where σ^{i} $(i=0,\ 1)$are world-sheet coordinates and $\Theta^{I}(I=1,\ 2)$ are two 32-component Majorana Weyl spinors of positive chirality.(The definition and some properties of Majorana spinors are summarized in Appendix.) The action is given by

$$S=\int d^2\sigma\left(-\frac{1}{2}\sqrt{-g}\,g^{ij}L_i^{\hat{a}}L_j^{\hat{b}}\,\eta_{\hat{a}\hat{b}}-2i\,\varepsilon^{ij}\int dt L_{it}^{\hat{a}}S^{IJ}\bar{\Theta}_I\Gamma^{\hat{a}}L_{jt}^{J}\right), \tag{3,6}$$

where s^{IJ} has non-vanishing components $s^{11}=-s^{22}=1$,

$$L=\left[\left(\frac{\sinh(tM)}{M}\right)D\Theta\right]^{I},$$

$$L_t^{\hat{a}}=dx^{\hat{m}}e_{\hat{m}}{}^{\hat{a}}-4i\bar{\Theta}^{I}\Gamma^{\hat{a}}\left[\left(\frac{\sinh^2(\frac{1}{2}tM)}{M^2}\right)D\,\Theta\right]^{I},$$

$$(M^2)^{IJ}=\varepsilon^{IK}(-\gamma^{A}\Theta^{K}\bar{\Theta}^{J}\gamma_{A}+\gamma^{a'}\Theta^{K}\bar{\Theta}^{J}\gamma_{a'})$$

$$+\frac{1}{2}\,\varepsilon^{KJ}(\gamma^{AB}\Theta^{I}\bar{\Theta}^{K}\gamma_{AB}-\gamma^{a'b'}\Theta^{I}\bar{\Theta}^{K}\gamma_{a'b'}),$$

$$(D\Theta)^{I}=\left[d+\frac{1}{4}(\omega^{AB}\gamma_{AB}+\omega^{a'b'}\gamma_{a'b'})\right]\Theta^{I}-\frac{1}{2}i\,\varepsilon^{IJ}(e^{A}\gamma_{A}+ie^{a'}\gamma^{a'})\Theta^{J}, \tag{3.7}$$

and $L^I = L_{t=I}$, $L^{\dot a} = L_{t=I}^{\dot a}$. This action is invariant under world-sheet reparametrizations and κ-transformations as well as transformations of the supergroup SU (2,2|4) containing the isometry SO (2,4) $\times$SO(6) of $AdS_5 \times S^5$ as a subgroup.

The gauge-fixed action of the type IIB string theory in $AdS_5 \times S^5$ was obtained in refs. [11]−[14]. The κ-symmetry was fixed by using Killing spinor gauge [11]

$$\Theta^2 = -i\gamma^4 \Theta^1. \tag{3.8}$$

In this gauge the action is greatly simplified. To simplify the expressions further we define new variables θ^I as

$$\Theta^I = \gamma^{\frac{1}{2}} f(x^{m'}) \begin{pmatrix} \theta^I \\ 0 \end{pmatrix}, \tag{3.9}$$

where θ^I have 16 components and

$$f(x^{m'}) = e^{\frac{1}{2} i x^9 \gamma_9 \gamma_4} e^{-\frac{1}{2} x^8 \gamma_{89}} e^{-\frac{1}{2} x^7 \gamma_{78}} e^{-\frac{1}{2} x^6 \gamma_{67}} e^{-\frac{1}{2} x^5 \gamma_{56}}, \tag{3.10}$$

is a quantity introduced in [17] as a solution of a differential equation

$$\left(\partial_{m'} + \frac{1}{4} \omega_{m'}{}^{a'b'} \gamma_{a'b'} - \frac{1}{2} i \gamma_{m'} \gamma_4 \right) f = 0. \tag{3.11}$$

The geuge condition (3.8) becomes $\theta^2 = -i\gamma^4 \theta^1$. L' s in eq. (3.7) are then simplified as

$$L_t^a = r(dx^m \delta_m^a - it^2 \theta^I \gamma^a d\theta^I),$$
$$L_t^4 = \frac{dr}{r},$$
$$L^{a'} = dx^{m'} e_{m'}{}^{a'} ,$$
$$L_t^I = t r^{\frac{1}{2}} f \begin{pmatrix} d\theta^I \\ 0 \end{pmatrix}. \tag{3.12}$$

Since the explicit form of the action in the coordinate system used in eq. (3.1) was not given in ref/. [12], we shall now work out it. To write down the second term (the Wess-Zumino term) of the action (3.6) we introduce

$$\Sigma^4 = \gamma^0 f^{-1} \gamma_0 \gamma^4 f, \qquad \Sigma^{a'} = \gamma^0 f^{-1} \gamma_0 \gamma^{a'} f. \tag{3.13}$$

From eq. (3.11) it can be shown that they satisfy

$$\partial_{m'} \Sigma^4 = i e_{m'}{}^{a'} \Sigma_{a'} ,$$
$$\partial_{m'} \Sigma^{a'} + \omega_{m'}{}^{a'}{}_{b'} \Sigma^{b'} = i e_{m'}{}^{a'} \Sigma^4. \tag{3.14}$$

Then, the Wess-Zumino term can be written as

$$S_{WZ} = \int d^2\sigma\,(-2i)\,\varepsilon^{ij}\,(\partial_i r\bar{\theta}\,\Sigma^4\partial_j\theta + i\partial_i x^{m'}\,r\bar{\theta}\,\Sigma_{m'}\,\partial_j\theta)$$

$$= \int d^2\sigma\,2i\,\varepsilon^{ij} r\,\partial_i\bar{\theta}\,\Sigma^4\partial_j\theta, \tag{3.15}$$

where $\theta \equiv \theta^1 = i\gamma^4\theta^2$ and we have used the first equation in eq. (3.14) to derive the second expression. Using the field equation of g_{ij} to eliminate it we obtain our final form of the action

$$S = \int d^2\sigma\,(-\sqrt{-h} + 2i\,\varepsilon^{ij} r\,\partial_i\bar{\theta}\,\Sigma^4\partial_j\theta), \tag{3.16}$$

where

$$h_{ij} = E_i^{\hat{a}} E_j^{\hat{b}}\,\eta_{\hat{a}\hat{b}} \tag{3.17}$$

with

$$E_i^a = r\,(\partial_i x^m\,\delta_m^a - 2i\bar{\theta}\,\gamma^a\partial_i\theta),\ E_i^4 = \frac{\partial_i r}{r},\ E_i^{a'} = \partial_i x^{m'} e_{m'}{}^{a'}. \tag{3.18}$$

4. Symmetries of the action

The action (3.6) is invariant under transformations of the supergroup SU (2, 2|4). The gauge-fixed action (3.16) should also has the SU (2, 2|4) symmetry. However, some of the original SU (2, 2|4) transformations do not preserve the gauge condition (3.8). To recover the gauge condition one has to add a compensating κ-transformation to the original SU (2, 2|4) transformation.

First consider a subgroup SO (2, 4) of SU (2, 2|4), which is the isometry of AdS_5. Before the gauge-fixing (3.8) the transformations of the bosonic variables are (2.7) and $\delta x^{m'} = 0$. The transformations of the fermionic variables are given by the local Lorentz transformations

$$\delta\Theta^I = \frac{1}{4}\lambda^{AB}\Gamma_{AB}\Theta^I \tag{4.1}$$

with the transformation parameters λ^{AB} in eq. (2.13). In terms variable θ these transformations become

(i) $\delta\theta = \frac{1}{4}a^{ab}\gamma_{ab}\theta,$

(ii) $\delta\theta = \frac{1}{2}\Lambda\theta$,

(iii) $\delta\theta = 0$,

(iv) $\delta\theta = -(x^a\zeta^b\gamma_a\gamma_b + \frac{1}{r}\zeta^a\gamma_a\Sigma^4)\theta$. (4.2)

The transformations (i)–(iii) preserve the gauge condition (3.8) and do not need a compensating κ-transformation. Indeed, it is easy to see that the gauge-fixed action (3.16) is invariant under these transformations. On the other hand, the transformation (iv) does not preserve the gauge condition and needs a compensating κ-transformation. Without this κ-transformation the gauge-fixed action is not invariant. We will discuss this transformation in the next section.

As for the supertransformations in SU (2, 2|4) we distinguish the cases $\varepsilon^2 = -i\gamma^4\varepsilon^1$ and $\varepsilon^2 = i\gamma^4\varepsilon^1$. For the case $\varepsilon^2 = -i\gamma^4\varepsilon^1$ the transformation preserves the gauge condition (3.8) and is given by

$$\delta x^m = 2i\bar{\varepsilon}\gamma^a\theta\delta^m_a, \quad \delta r = 0, \ \delta x^{m'} = 0, \quad \delta\theta = \varepsilon. \tag{4.3}$$

The gauge-fixed action is indeed invariant under this transformation. On the other hand, the case $\varepsilon^2 = i\gamma^4\varepsilon^1$ does not preserve the gauge condition (3.8) and needs a compensating κ-transformation. It is not easy to obtain an explicit form of this κ-transformation. directly. One way to obtain this transformation is to use the commutator algebra of the transformations. From the commutation relations of the SU (2, 2|4) algebra this transformation can be obtained from the commutator of the bosonic transformation (iv) and the supertransformation for $\varepsilon^2 = -i\gamma^4\varepsilon^1$. Therefore, once we know the form of the bosonic transformation (iv), which we will discuss in the next section, we can easily find the supertransformation for $\varepsilon^2 = i\gamma^4\varepsilon^1$.

5. Modified isometry transformations

In this section we try to obtain an explicit form of the bosonic transformation (iv), which needs a compensating κ-transformation. Under the original transformation (iv) in eqs. (2.7), (4.2) the variation of the Lagrangian up to quartic terms in θ is

$$\delta L = -2i(\sqrt{-h}h^{ij} - \varepsilon^{ij})(2r\,\partial_j x^a \zeta_a \bar{\theta}\Sigma^4 \partial_i \theta + 2r^{-1}\partial_j r\,\zeta^a \bar{\theta}\gamma_a \partial_i \theta$$
$$- \partial_j r\,\partial_i x^a \zeta^b \bar{\theta}\gamma_{ab}\Sigma^4\theta + ir\,\partial_j x^a \partial_i x^{m'} \zeta^b \bar{\theta}\gamma_{ab}\Sigma_{m'}\theta)$$
$$+ 4\varepsilon^{ij}\partial_j x^{m'} \zeta^a \bar{\theta}\Sigma^4 \gamma_a \Sigma_{m'} \partial_i \theta. \tag{5.1}$$

We see that $O(\theta^0)$ terms have completely cancelled out but $O(\theta^2)$ terms remain. Thus, the action is not invariant under this transformation.

We add extra terms δ' to the transformations (2.7), (4.2) of order

$$\delta' x^m,\ \delta' r,\ \delta' x^{m'} = O(\theta^2), \qquad \delta'\theta = O(\theta) \tag{5.2}$$

corresponding a compensating κ-transformation. In principle, δ' can be obtained by explicitly constructing a compensating κ-transformation. However, we choose another route to find them. We make a plausible guess about the transformations and check the invariance of the action under them. The variation of the Lagrangian up to quartic terms in θ under the new terms in the transformations is

$$\delta' L = \partial i(\sqrt{-h}h^{ij} r^2 \partial_j x^a)(\delta' x^a - 2i\bar{\theta}\gamma^a \delta'\theta) - 2\sqrt{-h}h^{ij}\,\partial_i x^{m'} D_j(\delta' x^{n'} g_{m'n'})$$
$$-\sqrt{-h}h^{ij}\left(\frac{\delta' r}{r} E_{ia}E_j^a + \delta' r\,\frac{\partial_i r\,\partial_j r}{r^3} + \frac{\partial_i r}{r^2}\partial_j \delta' r\right)$$
$$-4i\varepsilon^{ij} r\partial i\,\bar{\theta}\gamma^0 f^{-1}\gamma_0(E_j{}^A\gamma_A + iE_j{}^{a'}\gamma_{a'}) f\delta'\theta$$
$$-4i(\sqrt{-h}h^{ij} - \varepsilon^{ij})\, r^2 \partial_j x^a\, \partial_i \bar{\theta}\gamma_a \delta'\theta. \tag{5.3}$$

To find required transformations we introduce a matrix

$$\Gamma = \frac{1}{2\sqrt{-h}}\,\varepsilon^{ij} E_i{}^{\hat{a}} E_j{}^{\hat{b}} \Gamma_{\hat{a}\hat{b}}, \qquad \Gamma^2 = 1, \tag{5.4}$$

which satisfies an identity

$$\sqrt{-h}h^{ij} E_j{}^{\hat{a}} \Gamma_{\hat{a}} \Gamma = \varepsilon^{ij} E_j{}^{\hat{a}} \Gamma_{\hat{a}}. \tag{5.5}$$

In the representation (3.3) of the gamma matrices Γ becomes

$$\Gamma = \Gamma_0 \otimes 1 + \Gamma_1 \otimes i\sigma_3, \tag{5.6}$$

where

$$\Gamma_0 = \frac{1}{2\sqrt{-h}}\,\varepsilon^{ij}(E_i{}^A E_j{}^B \gamma_{AB} + E_i{}^{\dot{a}} E_j{}^{b}\gamma_{a'b'}),$$

$$\Gamma_1 = \frac{1}{\sqrt{-h}}\,\varepsilon^{ij} E_i{}^A E_j{}^{b'} \gamma_A \gamma_{b'}. \tag{5.7}$$

From $\Gamma^2=1$ we obtain

$$(\Gamma_0 \pm i\Gamma_1)^2=1. \tag{5.8}$$

The identity (5.5) is then written as

$$\sqrt{-h}h^{ij}(E_j{}^A\gamma_A \pm iE_j{}^{a'}\gamma_{a'})(\Gamma_0 \pm i\Gamma_1)=\varepsilon^{ij}(E_j{}^A\gamma_A \pm iE_j{}^{a'}\gamma_{a'}). \tag{5.9}$$

We may try a guess about the transformations. For instance we may try

$$\begin{aligned}
&\delta' x^m = 2i\bar\theta\gamma^a\delta'\theta\,\delta_a^m - 2ir^{-1}\zeta_b\bar\theta\gamma^{ab}\Sigma^4\theta\,\delta_a^m,\\
&\delta' r = 0,\\
&\delta' x^{m'} = \zeta^a\bar\theta\Sigma^4\gamma_a\Sigma^{m'}\theta,\\
&\delta'\theta = r^{-1}[1-f^{-1}(\Gamma_0+i\Gamma_1)f]\,\zeta^a\gamma_a\Sigma_4\theta.
\end{aligned} \tag{5.10}$$

There are many cancellations in the variation of the Lagrangian under this transformation due to the identity (5.9). There remain only

$$(\delta+\delta')L=4i(\sqrt{-h}h^{ij}-\varepsilon^{ij})\,r\,\partial_j x^a\zeta^b\partial_i\bar\theta\gamma_a[1+f^{-1}(\Gamma_0+i\Gamma_1)f]\gamma_b\Sigma^4\theta. \tag{5.11}$$

There terms, however, do not cancel. We have to look for a transformation under which the action is invariant but have not succeeded yet.

Appendix: Identities for Majorana spinors

A Majorana spinor ψ satisfies

$$\bar\psi=-\psi^T CC', \qquad \psi=-C^{-1}C'^{-1}\bar\psi^T, \tag{A.1}$$

where C and C' are the charge conjugation matrices of SO (1, 4) and SO(5) respectively. They satisfy

$$C\gamma^{a1\cdots an}C^{-1}=-\varepsilon_n(\gamma^{a1\cdots an})^T, \quad C'\gamma^{a'1\cdots a'n}C'^{-1}=-\varepsilon_n(\gamma^{a'1\cdots a'n})^T, \tag{A.2}$$

where

$$\varepsilon_n=\begin{cases}+1 & (n=2,\,3)\\ -1 & (n=0,\,1,\,4,\,5).\end{cases} \tag{A.3}$$

From eqs. (A.1), (A.3) we obtain for Majorana spinors ψ and χ

$$\bar\psi\gamma^{a1\cdots am}\gamma^{b'1\cdots b'n}\chi=-\varepsilon_m\varepsilon_n\bar\chi\gamma^{a1\cdots am}\gamma^{b'1\cdots b'n}\psi. \tag{A.4}$$

For the quantities defined in eq.(3.13) we obtain

$$CC'\Sigma^4C^{-1}C'^{-1}=(\Sigma^4)^T, \qquad CC'\Sigma^{a'}C^{-1}C'^{-1}=(\Sigma^{a'})^T, \tag{A.5}$$

and

$$\bar{\psi}\Sigma^4\chi = -\bar{\chi}\Sigma^4\psi, \qquad \bar{\psi}\Sigma^{a'}\chi = \bar{\chi}\Sigma^{a'}\psi. \tag{A.6}$$

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